Some New Trigonometric, Hyperbolic and Exponential Measures of Intutionistic Fuzzy Information.

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Abstract: New Trigonometric, Hyperbolic and Exponential Measures of Intutionistic Fuzzy Entropy and Intutionistic Fuzzy Directed Divergence are obtained and some particular cases have been discussed.

Index Terms: Intutionistic Fuzzy Set, Intutionistic Fuzzy Entropy, Intutionistic Fuzzy Directed Divergence, Measures of Intutionistic Fuzzy Information.

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1. Introduction:

Uncertainty and fuzziness are the basic nature of human thinking and of many real world objectives. Fuzziness is found in our decision, in our language and in the way we process information. The main use of information is to remove uncertainty and fuzziness. In fact, we measure information supplied by the amount of probabilistic uncertainty removed in an experiment and the measure of uncertainty removed is also called as a measure of information while measure of fuzziness is the measure of vagueness and ambiguity of uncertainties. Shannon [2] used "entropy" to measure uncertain degree of the randomness in a probability distribution. Let X is a discrete random variable with probability distribution $P = (p_{1,}p_{2,} \dots\dots, p_n)$ in an experiment. The information contained in this experiment is given by

$$H(P) = -\sum_{i=1}^{n} p_i log p_i \qquad (1)$$

Which is well known Shannon entropy.

The concept of entropy has been widely used in different areas, e.g. communication theory, statistical mechanics, finance, pattern recognition, and neural network etc. Fuzzy set theory developed by Lofti A. Zadeh [8] has found wide applications in many areas of science and technology, e.g. clustering, image processing, decision making etc. because of its capability to model non-statistical imprecision or vague concepts.

It may be recalled that a fuzzy subset A in U (universe of discourse) is characterized by a *membership function* $\mu_A: U \to [0,1]$ which represents the *grade of membership* of $x \in U$ *in* A as follows

$$\mu_A(x) = 0 \; if\; x\; does\; not\; belongs\; to\; A, \; and\; there\; is\; no\; uncertainty$$

$$= 1 \; if\; x\; belongs\; to\; A\; and\; there\; is\; no\; uncertainty$$

$$= 0.5 \; if\; maximum\; uncertainty$$

In fact $\mu_A(x)$ associates with each $x \in U$ a grade of membership in the set *A*. When $\mu_A(x)$ is valued in $\{0,1\}$ it is the characteristic function of a crisp (i.e. nonfuzzy) set. Since $\mu_A(x)$ and $1 - \mu_A(x)$ gives the same degree of fuzziness, therefore, corresponding to the entropy due to Shannon [2], De Luca and Termini [1] suggested the following measure of fuzzy entropy:

$$H(A) = -\left[\sum_{i=1}^{n} \mu_A(x_i) log \mu_A(x_i) + \sum_{i=1}^{n} (1 - \mu_A(x_i)) log(1 - \mu_A(x_i))\right] \qquad (2)$$

De Luca and Termini introduced a set of properties and these properties are widely accepted as a criterion for defining any new fuzzy entropy. In fuzzy set theory, the entropy is a measure of fuzziness which expresses the amount of average ambiguity/difficulty in making a decision whether an element belongs to a set or not. So, a measure of average fuzziness in a fuzzy set should have at least the following properties to be valid fuzzy entropy:

i) $H(A) = 0$ when $\mu_A(x_i) = 0 \; or \; 1$.
ii) $H(A)$ increases as $\mu_A(x_i)$ increases from 0 to 0.5.
iii) $H(A)$ decreases as $\mu_A(x_i)$ increases from 0.5 to 1.
iv) $H(A) = H(\bar{A})$, i.e. $\mu_A(x_i) = 1 - \mu_A(x_i)$

v) $H(A)$ is a concave function of $\mu_A(x_i)$.

Kullback and Leibler [7] obtained the measure of directed divergence of probability distribution $P = (p_1, p_2, \ldots, p_n)$ from the probability distribution $Q = (q_1, q_2, \ldots, q_n)$ as

$$D(P:Q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} \quad (3)$$

Let A and B be two standard fuzzy sets with same supporting points $x_1, x_2, \ldots, x_n$ and with fuzzy vectors $\mu_A(x_1), \mu_A(x_2), \ldots, \mu_A(x_n)$ and $\mu_B(x_1), \mu_B(x_2), \ldots, \mu_B(x_n)$. The simplest measure of fuzzy directed divergence as suggested by Bhandari and Pal (1993), is

$$D(A:B) = \sum_{i=1}^{n} \mu_A(x_i) \log \frac{\mu_A(x_i)}{\mu_B(x_i)}$$

$$+ \sum_{i=1}^{n} (1 - \mu_A(x_i)) \log \frac{(1 - \mu_A(x_i))}{(1 - \mu_B(x_i))} \quad (4)$$

satisfying the conditions:

i) $D(A:B) \geq 0$
ii) $D(A:B) = 0 \; iff \; A = B$
iii) $D(A:B) = D(B:A)$
iv) $D(A:B)$ is a convex function of $\mu_A(x_i)$

later kapur [5],[6] introduced a number of trigonometric hyperbolic and exponential measures of fuzzy entropy and fuzzy directed divergence. In section 2 and 3 we introduce some new trigonometric, hyperbolic and exponential measures of intutionistic fuzzy entropy and measures of intutionistic fuzzy directed divergence. Now we define the concept of intutionistic fuzzy set and then discuss the properties of intutionistic fuzzy entropy and intutionistic fuzzy directed divergence.

The concept of intutionistic fuzzy set was first time given by K. T. Atanasav (1983) as

Intutionistic fuzzy set: - Let a set E be fixed. An intutionistic fuzzy set (IFs) A of E is an object having the form $A = \{< x, \mu_A(x), v_a(x) > x \in E\}$ where the function $\mu_A: E \to [0,1]$ and $v_A: E \to [0,1]$ define respectively the degree of membership and degree of non membership of the element $x \in E$ to the set A, which is a subset of E and for every $x \in E$, $0 \leq \mu_A(x) + v_A(x) \leq 1$.

Conditions for measures of intutionistic fuzzy entropy:-

1. It should be defined in the range $0 \leq \mu_A(x) + v_A(x) \leq 1$
2. It should be continuous in this range.
3. It should be zero when $\mu_A(x) = 0$ and $v_A(x) = 1$.
4. It should not be changed when $\mu_A(x)$ changed in to $v_A(x)$.
5. It should be increasing function of $\mu_A(x)$ in the range $0 \leq \mu_A(x) \leq 0.5$ and decreasing function of $v_A(x)$ in the range $0 \leq v_A(x) \leq 0.5$
6. It should be concave function of $\mu_A(x)$.

Conditions for measures of intutionistic fuzzy directed divergence:- Let A and B be two standard intutionistic fuzzy sets with same supporting points $x_1, x_2, \ldots, x_n$ with membership $\mu_A(x_1), \mu_A(x_2), \ldots, \mu_A(x_n)$ and $\mu_B(x_1), \mu_B(x_2), \ldots, \mu_B(x_n)$. and non membership $v_A(x_1), v_A(x_2), \ldots, v_A(x_n)$ and $v_B(x_1), v_B(x_2), \ldots, v_B(x_n)$.The measure of intutionistic fuzzy directed divergence is denoted by $D(A:B)$ satisfying the following conditions.

i) $D(A:B) \geq 0$
ii) $D(A:B) = 0 \; iff \; A = B$
iii) $D(A:B) = D(B:A)$
iv) $D(A:B)$ is a convex function of $\mu_A(x_i)$.

2. New Measures of Intutionistic Fuzzy Entropy

2.1 Trigonometric Measure of Intutionistic Fuzzy Entropy

Consider the function $sin\pi x$ where $0 \leq x \leq 1$, is a convex function which gives us

$$H_1(A) = \sum_{i=1}^{n} sin(\pi \mu_A(x_i))$$

$$+ \sum_{i=1}^{n} sin(\pi v_A(x_i)) \quad (5)$$

is a new measure of intutionistic fuzzy entropy.

Clearly (5) is defined in the range $0 \leq \mu_A(x) + v_A(x) \leq 1$. It is continous in this range. It is zero when $\mu_A(x) = 0$ and $v_A(x) = 1$. It does not changed when $\mu_A(x)$ changed in to $v_A(x)$. It is an increasing function of $\mu_A(x)$ in the range $0 \leq \mu_A(x) \leq 0.5$ and decreasing function of $v_A(x)$ in the range $0 \leq v_A(x) \leq 0.5$. Let us consider $\mu_A(x) = s \text{ and } v_A(x) = c(constant)$ then $f(s) = sin\pi s + sin\pi c$ differentiating twice w.r.t. s we get $f'(s) = \pi(cos\pi s)$ and $f''(s) = -\pi^2(sin\pi s) < 0$. Therefore $H_1(A)$ is a concave function of $\mu_A(x)$. So (5) is a valid measure of intutionistic fuzzy entropy.

in particular for $\beta \leq \pi$

$$H_2(A) = \sum_{i=1}^{n} sin(\beta\mu_A(x_i)) + \sum_{i=1}^{n} sin(\beta v_A(x_i)) - sin\beta \quad (6)$$

is also a new measure of intutionistic fuzzy entropy.

(5) is a special case of (6) when $\beta = \pi$.

Another special case of (6) arises when $\beta = \frac{\pi}{2}$ we get

$$H_3(A) = \sum_{i=1}^{n} sin\left(\frac{\pi}{2}\mu_A(x_i)\right) + \sum_{i=1}^{n} sin\left(\frac{\pi}{2}v_A(x_i)\right) - 1 \quad (7)$$

Another trigonometric measure of intutionistic fuzzy entropy is

$$H_4(A) = \sum_{i=1}^{n} sin(\beta\mu_A(x_i) + \alpha) + \sum_{i=1}^{n} sin(\beta v_A(x_i) + \alpha) - sin(\alpha + \beta) \quad (8)$$

(8) reduces to (6) when $\alpha = 0$.

(8) reduces to (7) when $\alpha = 0, \beta = \frac{\pi}{2}$.

(8) reduces to (5) when $\alpha = 0, \beta = \pi$.

(8) is a 2-parameter measure of intutionistic fuzzy entropy.

If we put $\propto = \frac{\pi}{2}$ we get

$$H_5(A) = \sum_{i=1}^{n} cos(\beta\mu_A(x_i)) + \sum_{i=1}^{n} cos(\beta v_A(x_i)) - cos\beta \quad (9)$$

is a new measure of intutionistic fuzzy entropy. Clearly above given measures of intutionistic fuzzy entropies are satisfying all the properties which are given in section 1. So these are valid measures of intutionistic fuzzy entropy.

2.1 Hyperbolic Measure of Intutionistic Fuzzy Entropy

$sinhx, coshx, tanhx$ where $0 \leq x \leq 1$ are all convex functions and gives us following valid measures of intutionistic fuzzy entropy

$$H_6(A) = sinh\beta - \sum_{i=1}^{n} sinh(\beta\mu_A(x_i)) - \sum_{i=1}^{n} sinh\beta(v_A(x_i)) \quad (10)$$

$$H_7(A) = cosh\beta - \sum_{i=1}^{n} cosh(\beta\mu_A(x_i)) - \sum_{i=1}^{n} cosh\beta(v_A(x_i)) \quad (11)$$

$$H_8(A) = tanh\beta - \sum_{i=1}^{n} tanh(\beta\mu_A(x_i)) - \sum_{i=1}^{n} tanh(\beta v_A(x_i)) \quad (12)$$

Since $x^m sinhx, x^m coshx, x^m tanhx$ are also convex functions for $m \geq 1$, we get the following additional measures of intutionistic fuzzy entropy.

$$H_9(A) = sinh\beta - \sum_{i=1}^{n} \mu_A{}^m(x_i) sinh(\beta\mu_A(x_i)) - \sum_{i=1}^{n} v_A{}^m(x_i) sinh(\beta v_A(x_i)) \quad (13)$$

$$H_{10}(A) = cosh\beta - \sum_{i=1}^{n} \mu_A{}^m(x_i)cosh(\beta\mu_A(x_i))$$

$$- \sum_{i=1}^{n} v_A{}^m(x_i)cosh(\beta v_A(x_i)) \quad (14)$$

$$H_{11}(A) = tanh\beta - \sum_{i=1}^{n} \mu_A{}^m(x_i)tanh(\beta\mu_A(x_i))$$

$$- \sum_{i=1}^{n} v_A{}^m(x_i)tanh(\beta v_A(x_i)) \quad (15)$$

2.2 Exponential Measures of Intutionistic Fuzzy Entropy

Since $x^m e^{ax}$ is a convex function when $m \geq 1, x > 0$ we get the measure of intutionistic fuzzy entropy

$$H_{12}(A) = e^a - \sum_{i=1}^{n} \mu_A{}^m(x_i)\, e^{a\mu_A(x_i)}$$
$$- \sum_{i=1}^{n} v_A{}^m(x_i)\, e^{av_A(x_i)} \quad (16)$$

3. New Measures of Intutionistic Fuzzy Directed Divergence

3.1 New Hyperbolic Measures of Intutionistic Fuzzy Directed Divergence

Using the convexity of $sinhx, coshx, tanhx$ we get the following measures of hyperbolic intutionistic fuzzy directed divergence.

$$D_1(A{:}B) = \sum_{i=1}^{n} \mu_B(x_i)\, sinh\left(\beta \frac{\mu_A(x_i)}{\mu_B(x_i)}\right)$$

$$+ \sum_{i=1}^{n} v_B(x_i)\, sinh\left(\beta \frac{v_A(x_i)}{v_B(x_i)}\right)$$

$$- \sum_{i=1}^{n} \mu_B(x_i)sinh\beta - \sum_{i=1}^{n} v_B(x_i)sinh\beta \quad (17)$$

Clearly

i) $D_1(A{:}B) \geq 0$
ii) $D_1(A{:}B) = 0\ iff\ A = B$
iii) $D_1(A{:}B) = D_1(B{:}A)$
iv) $D_1(A{:}B)$ is a convex function $\mu_A(x_i)$.

$D_1(A{:}B)$ satisfies all properties of intutionistic fuzzy directed divergence and therefore $D_1(A{:}B)$ is a valid measure of intutionistic fuzzy directed divergence.

$$D_2(A{:}B) = \sum_{i=1}^{n} \mu_B(x_i)\, cosh\left(\beta \frac{\mu_A(x_i)}{\mu_B(x_i)}\right)$$

$$+ \sum_{i=1}^{n} v_B(x_i)\, cosh\left(\beta \frac{v_A(x_i)}{v_B(x_i)}\right)$$

$$- \sum_{i=1}^{n} \mu_B(x_i)cosh\beta - \sum_{i=1}^{n} v_B(x_i)cosh\beta \quad (18)$$

$$D_3(A{:}B) = \sum_{i=1}^{n} \mu_B(x_i)\, tanh\left(\beta \frac{\mu_A(x_i)}{\mu_B(x_i)}\right)$$

$$+ \sum_{i=1}^{n} v_B(x_i)\, tanh\left(\beta \frac{v_A(x_i)}{v_B(x_i)}\right)$$

$$- \sum_{i=1}^{n} \mu_B(x_i)tanh\beta - \sum_{i=1}^{n} v_B(x_i)tanh\beta \quad (19)$$

Again since $x^m sinhx, x^m coshx, x^m tanhx$ are also convex functions for $m \geq 1$, we get the following more general hyperbolic measures of intutionistic fuzzy directed divergence.

$$D_4(A{:}B) = \sum_{i=1}^{n} \mu_A{}^m(x_i)\mu_B{}^{1-m}(x_i)\, sinh\left(\beta \frac{\mu_A(x_i)}{\mu_B(x_i)}\right)$$

$$+ \sum_{i=1}^{n} v_A{}^m(x_i)v_B{}^{1-m}(x_i)\, sinh\left(\beta \frac{v_A(x_i)}{v_B(x_i)}\right)$$

$$- \sum_{i=1}^{n} \mu_B(x_i)sinh\beta - \sum_{i=1}^{n} v_B(x_i)sinh\beta \quad (20)$$

$$D_5(A{:}B) = \sum_{i=1}^{n} \mu_A{}^m(x_i)\mu_B{}^{1-m}(x_i)\, cosh\left(\beta \frac{\mu_A(x_i)}{\mu_B(x_i)}\right)$$

$$+ \sum_{i=1}^{n} v_A{}^m(x_i)v_B{}^{1-m}(x_i)\, cosh\left(\beta \frac{v_A(x_i)}{v_B(x_i)}\right)$$

$$- \sum_{i=1}^{n} \mu_B(x_i)cosh\beta - \sum_{i=1}^{n} v_B(x_i)cosh\beta$$

(21)

$$D_6(A:B) = \sum_{i=1}^{n} \mu_A{}^m(x_i)\mu_B{}^{1-m}(x_i)\tanh\left(\beta\frac{\mu_A(x_i)}{\mu_B(x_i)}\right) + \sum_{i=1}^{n} v_A{}^m(x_i)v_B{}^{1-m}(x_i)\tanh\left(\beta\frac{v_A(x_i)}{v_B(x_i)}\right) - \sum_{i=1}^{n}\mu_B(x_i)\tanh\beta - \sum_{i=1}^{n} v_B(x_i)\tanh\beta \quad (22)$$

3.2 New Exponential Measures of Fuzzy Intutionistic Directed Divergence

Since $x^m e^{ax}$ is a convex function when $m \geq 1, x > 0$ we get the following measures of intutionistic fuzzy directed divergence

$$D_7(A:B) = \sum_{i=1}^{n} \mu_A{}^m(x_i)\mu_B{}^{1-m}(x_i)\, e^{a\left(\frac{\mu_A(x_i)}{\mu_B(x_i)}\right)} + \sum_{i=1}^{n} v_A{}^m(x_i)v_B{}^{1-m}(x_i)\, e^{a\left(\frac{v_A(x_i)}{v_B(x_i)}\right)} - e^a \quad (23)$$

Special case for m=0 and m=1 are

$$D_8(A:B) = \sum_{i=1}^{n} \mu_B(x_i)\, e^{a\left(\frac{\mu_A(x_i)}{\mu_B(x_i)}\right)} + \sum_{i=1}^{n} v_B(x_i)\, e^{a\left(\frac{v_A(x_i)}{v_B(x_i)}\right)} - e^a \quad (24)$$

$$D_9(A:B) = \sum_{i=1}^{n} \mu_A(x_i)\, e^{a\left(\frac{\mu_A(x_i)}{\mu_B(x_i)}\right)} + \sum_{i=1}^{n} v_A(x_i)\, e^{a\left(\frac{v_A(x_i)}{v_B(x_i)}\right)} - e^a \quad (25)$$

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4. Conclusion

In section 2 and 3 by using the convexity of some trigonometric, hyperbolic and exponential function and satisfying the conditions of intutionistic fuzzy entropy and intutionistic fuzzy directed divergence we get some new trigonometric, hyperbolic and exponential measures of intutionistic fuzzy entropy and intutionistic fuzzy directed divergence.

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